St Leonard's College

Mathematical Methods Unit 4 2017

School Assessed Coursework: Problem Solving Task 1

Name: ANSWERS TASK 1

This task is based on concepts from the Differential and Integral Calculus areas of study of the course.
It consists of two parts, to be completed over 2 periods.

Reading Time: 5 minutes
Task Time: 70 minutes

/33.

Each part must be completed within a period and submitted at the conclusion of that period.
Students can refer to their bound reference material and calculator use is encouraged throughout the task, but no discussion is permitted between students during the task. Bound reference material may be checked by staff at any stage.
You can have access to this sheet throughout the task.

You will be provided with a sheet of miscellaneous formulae.
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This SAC contributes 8.5% towards the overall study score for Mathematical Methods.

Students will be required to demonstrate the achievement of the following outcomes:

Outcome 1	to define & explain key concepts and apply a range of related mathematical routines and procedures
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The overall result for this task will be comprised of marks for each of the above outcomes in the approximate proportions:

OUTCOME 1: 40%
OUTCOME 2: 38%
OUTCOME 3: 22%

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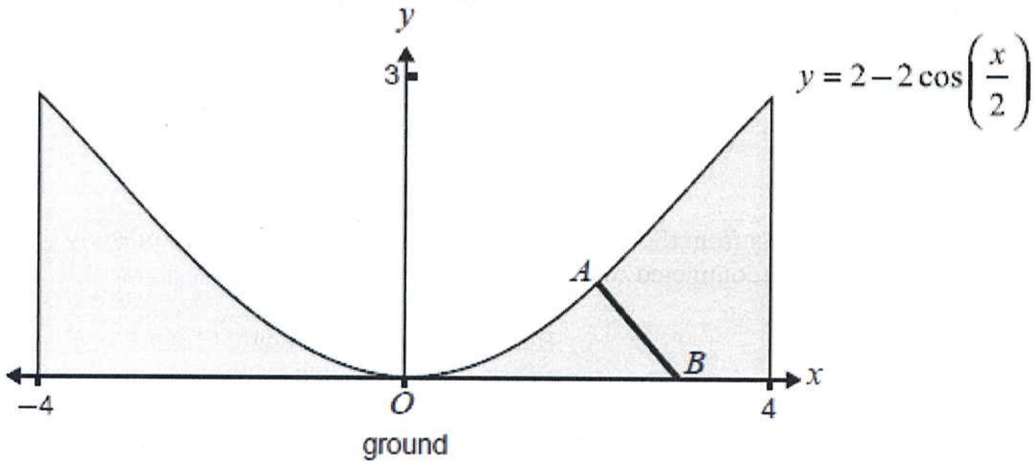
Question 1

Pythag is making a skateboard ramp. He draws a cross-section diagram with coordinate axes as shown below.



The curve has the equation $y = 2 - 2\cos\left(\frac{x}{2}\right)$, $-4 \le x \le 4$. All measurements are in metres; the horizontal length of the structure is 8 metres.

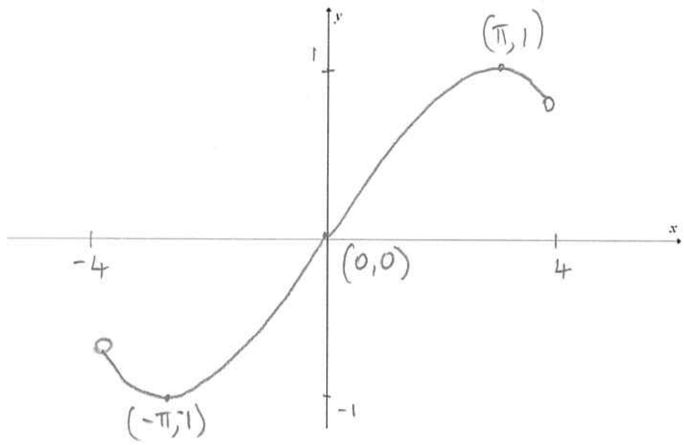
a. How many metres above the ground is the highest point of the ramp? Give your answer correct to two decimal places.

$f(4) = 2.83$

$\therefore$ height $= 2.83$m

[1 mark]

b. **i.** Sketch the graph of the gradient of the ramp, showing the coordinates of any axial intercepts and turning points.



(1) - end points

(1) - T.P. + labelled origin

(1) shape

[3 marks]

ii. Hence, state the range of the gradient of the ramp.

[-1, 1]

[1 mark]

/5

iii. At what horizontal distance(s) from O is the ramp the steepest?

at $-\pi$ and π

ok answer could be just $\underline{\underline{\pi}}$

[1 mark]

There is a supporting beam AB on the structure as shown. A is a point on the curve one metre vertically above the x-axis. B is a point on the x-axis such that AB is normal to the curve at A.

c. i. Find the value of the x-coordinate of A.

when $y=1$, $1 = 2-2\cos\left(\frac{x}{2}\right)$ (1) eqn $=1$

$x = \frac{2\pi}{3}$ (1) soln.

[2 marks]

ii. Find the equation of the normal to the curve at A.

$y = -\frac{2\sqrt{3}}{3}x + \frac{4\pi\sqrt{3}}{9} + 1$

[1 marks]

iii. Find the length of the beam AB.

$A: \left(\frac{2\pi}{3}, 1\right)$ and $B: \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}, 0\right)$ from $0 = -\frac{2\sqrt{3}}{3}x + \frac{4\pi\sqrt{3}+9}{9}$

$x = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

$\therefore$ distance $AB = \sqrt{1^2 + \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \frac{2\pi}{3}\right)^2}$

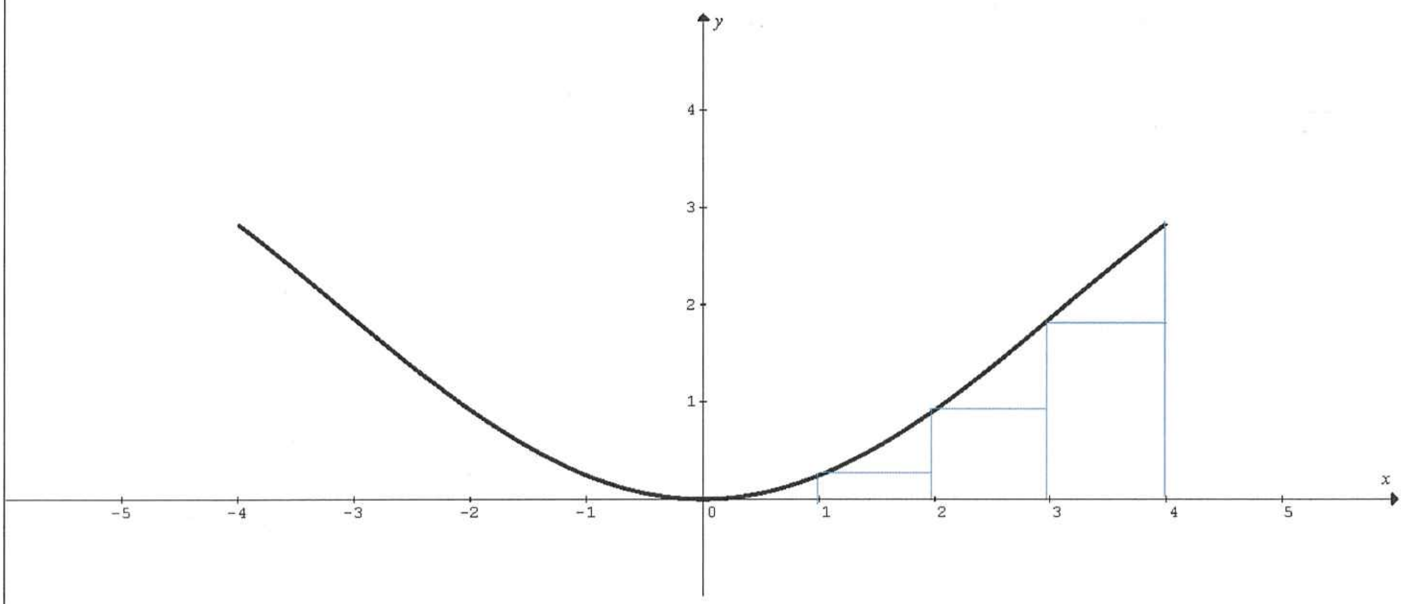
$= \sqrt{1+\frac{3}{4}}$ (1) using distance formula

$= \frac{\sqrt{7}}{2}$. (1) ANS

[2 marks]

6

Pythag wishes to erect boards to cover the front (cross section) of the ramp. He decides to estimate the area from $x = 0$ to $x = 4$ using rectangles of width 1 metre as shown below.



d. **i.** Complete the table of values for y, where $y = 2 - 2\cos\left(\frac{x}{2}\right)$, $-4 \le x \le 4$. giving values correct to two decimal places.

x	1	2	3	4
y	0.24	0.92	1.86	2.83

[1 mark]

ii. Use the table and the graph to find Pythag's approximation of the **total** area of the boards from $x = -4$ to $x = 4$, measured in square metres, correct to one decimal place.

$$A = 2 \times (0.24 + 0.92 + 1.86)$$

$$= 6.0 \text{ m}^2$$

[1 mark]

e. Use calculus to find the actual area of the shaded region, correct to two decimal places.

$$A = \int_{-4}^{4} 2 - 2\cos\left(\frac{x}{2}\right).dx$$

$$= 8.73 \text{ m}^2$$

[2 marks]

4

Question 2

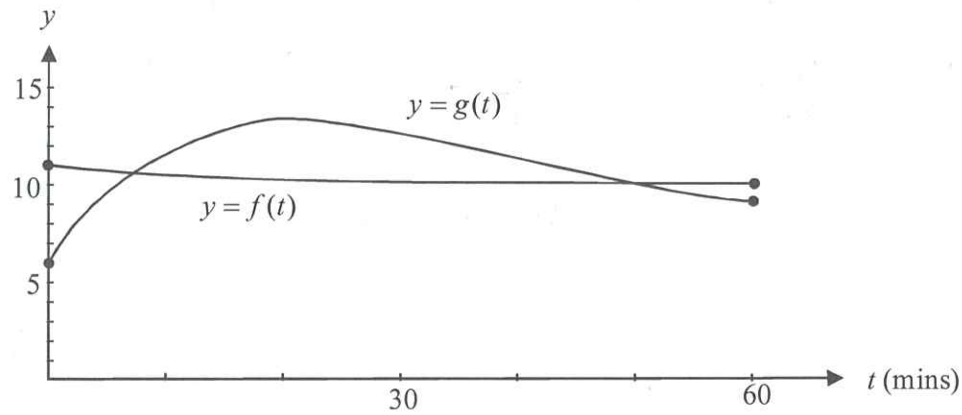
Pythag also enjoys investing in the stock market. The value of two stocks, Foolsgold and Gold Inc., are modelled respectively by the functions

$$f:[0,60] \to R,\ f(t) = e^{-\frac{t}{20}} + 10$$

$$\text{and } g:[0,60] \to R,\ g(t) = te^{-\frac{t}{20}} + 6$$

where f and g represent the value of the respective stocks, in dollars, t minutes after the opening of trade on a particular day.

The graphs of the functions are shown below.



a. From what period of time was the value of Gold Inc. stock greater than or equal to the value of Foolsgold stock? Give your answers correct to two decimal places.

$t_1 = 6.550$ and $t_2 = 50.186$ (1) finding $t_1 + t_2$

period of time $= 43.637$ minutes (11) ans.

≈ 43.64 minutes.

OR from 6.55 mins to 50.19 mins — 1 for each value. [2 marks]

b. What is the average rate of change (in dollars per minute) of Foolsgold stock in the first hour?

$f(0) = 11$ (1) seeing gradient calculation

$f(60) = e^{-3} + 10$

av. rate of change $= \dfrac{e^{-3} + 10 - 11}{60}$ (1) ans.

$= \$\dfrac{e^{-3} - 1}{60}$ per min

[2 marks]

c. Find the average value of the Gold Inc. stock during the first hour of trade. Give your answer correct to the nearest cent.

ave value $= \frac{1}{60}\int_0^{60} t.e^{-\frac{t}{20}} + 6 \, .dt$

(1) seeing the integral

(1) ans

$= \$11 \cdot 34$

[2 marks]

d. During the period when the value of the Gold Inc. stock was greater than the value of the Foolsgold stock, find the value of t when the difference in the values was a maximum.

difference (d) $= g(t) - f(t) = (t-1)e^{-\frac{t}{20}} - 4$

(1) eqn for difference

for max diff $\frac{d(d)}{dt} = 0 = \left(\frac{21}{20} - \frac{t}{20}\right)e^{-\frac{t}{20}}$

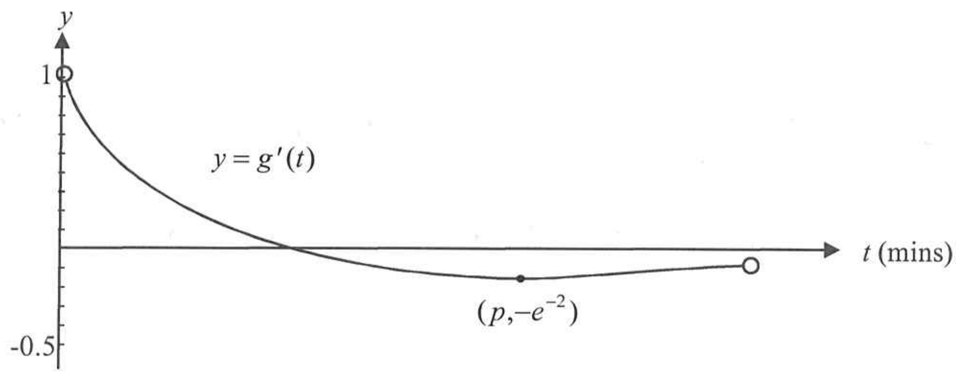
(1) derivative $= 0$

(1) ans

$t = 21$ minutes

[3 marks]

The graph of the derivative function $g'(t) = \left(1 - \frac{t}{20}\right)e^{-\frac{t}{20}}$, $t \in (0,60)$ is shown below. The graph has a minimum turning point at the point $(p, -e^{-2})$ where p is a positive integer.



e. **i.** Find the value of p.

$p = 40$

[1 mark]

ii. Hence find the value of the Gold Inc. stock when the rate at which it was decreasing was a maximum. Give your answer to the nearest cent.

$g(40) = \$11 \cdot 41$

[1 mark]

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Question 3

a. Pythag's daughter, Phaedra, enjoys drawing functions. She begins with the function $f: R^+ \to R,\ f(x) = \log_e(x)$. She then reflects it in both axes and translates it a ($a \in R^+$) units to the right. Explain how the rule of this transformed graph is given by $g(x) = -\log_e(a-x)$.

- $f(x) = -\log_e x$ reflection in x-axis
- $f_1(x) = -\log_e(-x)$ reflection in y-axis
- $f_2(x) = -\log_e(-(x-a))$ translation a units to the right

$\therefore\ g(x) = -\log_e(a-x)$

(1) reflection in x + y axes

(1) translation

*1 gave $\frac{1}{2}$ if they mentioned translation 1st.

[2 marks]

b. If this new function is $g: (-\infty, a) \to R,\ g(x) = -\log_e(a-x)$, where a is a positive real number, find, in terms of a, the x-coordinates of the points of intersection of the graphs of f and g.

$$-\log_e(a-x) = \log_e x$$

$$(a-x)^{-1} = x$$

$$\frac{1}{a-x} = x$$

$$1 = x(a-x)$$

$$x^2 - xa + 1 = 0$$

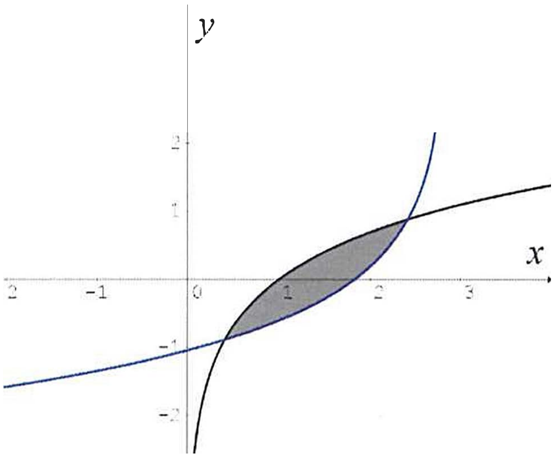
$$x = \frac{a \pm \sqrt{a^2-4}}{2}$$

(1) eqn stated correctly

(1) ans (s)

[2 marks]

c. Phaedra decides to let $a = 2\sqrt{2}$ and draws both $y = f(x)$ and $y = g(x)$ on the same set of axes as shown below.



i. Write down the integral that would find the shaded area between the two graphs.

when $a = 2\sqrt{2}$

$x = \dfrac{2\sqrt{2} \pm \sqrt{4}}{2} = \sqrt{2} \pm 1$

(1) finding x values when $a = 2\sqrt{2}$

$A = \int_{\sqrt{2}-1}^{\sqrt{2}+1} \log_e x + \log_e(2\sqrt{2} - x)$

(1) Correct integral

ii. Calculate this area giving your answer correct to 2 decimal places.

$A = 0.99$ sq units

(1) correct answer

[2 + 1 = 3 marks]

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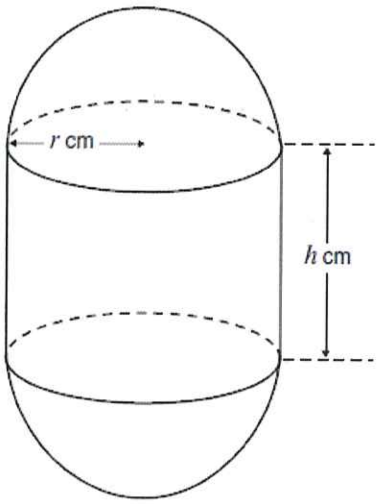
Question 1

Euler wants to construct a time capsule in which to bury some of his precious calculations and notes. The time capsule will be a right circular cylinder of height h cm, and radius r cm, with hemispherical caps of radius r cm on each end, as shown in the diagram.

Let the total volume of the capsule be V cm³.



a. Express V in terms of r and h.

$$V = \pi r^2 h + \frac{4}{3}\pi r^3$$

[1 mark]

b. The total volume of the capsule will be 8000 cm³.

i. Show that $h = \dfrac{8000}{\pi r^2} - \dfrac{4r}{3}$

$$8000 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\pi r^2 h = 8000 - \frac{4}{3}\pi r^3$$

$$h = \frac{8000}{\pi r^2} - \frac{4\pi r^3}{3\pi r^2} = \frac{8000}{\pi r^2} - \frac{4r}{3}$$

(1) Equating V=8000

(1) correct Method in rearranging eqn

[2 marks]

ii. The values which r may take lie in an interval. Find the end points of this interval, correct to two decimal places.

$$\frac{8000}{\pi r^2} - \frac{4r}{3} > 0$$

$$0 < r < 12.41$$

(1) for each end point.

[2 marks]

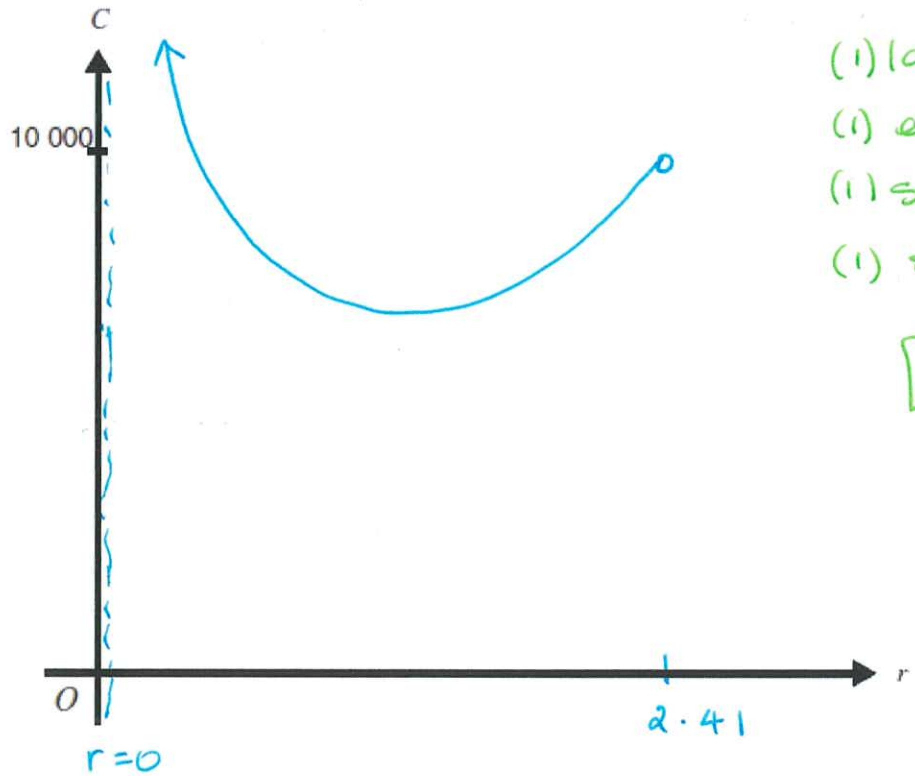
c. The material for the cylindrical part of the capsule costs 2 cents per cm^2 of surface. The material for the hemispherical caps costs 3 cents per cm^2 of surface. [The surface area of a sphere of radius r is $4\pi r^2$.]
Find an expression for C cents, the total cost of the materials for the capsule, in terms of *r*.

$$C = 2 \times 2\pi r h + 3 \times 4\pi r^2$$
$$= 4\pi r\left(\frac{8000}{\pi r^2} - \frac{4r}{3} + 3r\right)$$
$$= 4\pi r\left(\frac{8000}{\pi r^2} + \frac{5r}{3}\right)$$

(1) Correct eqn
(1) simplified

[2 marks]

d. Sketch the graph of C over an appropriate domain on the axes below. Label any horizontal or vertical asymptote with its equation.
You are not required to show the co-ordinates of any turning point.



(1) label asymptote
(1) end point
(1) scale on x-axis
(1) shape.

[-1 for any of above that is not present]

[3 marks]

e. Find the value of *r*, correct to two decimal places, for which *C* is a minimum.

$r = 9.14$

1

[~~2~~ marks]

Question 2

Galileo is testing a model train running on a on a linear track. The acceleration, a m s^{-2}, of Galileo's model train at time t seconds is given by $a = 2t + \cos(t)$.

i. Find the acceleration of the train at $t = 0$.

$a = \cos(0)$

$= 1 \text{ ms}^{-2}$

[1 mark]

ii. Show that the velocity, v m s^{-1}, at time t seconds, is given by $v = t^2 + \sin(t) - 3$ if the initial velocity of the train is -3 m s^{-1}.

$v = \int 2t + \cos(t) . dt$

$= t^2 + \sin t + c$

when $t = 0, \quad v = -3$

$-3 = 0 + 0 + c$

$\therefore c = -3.$

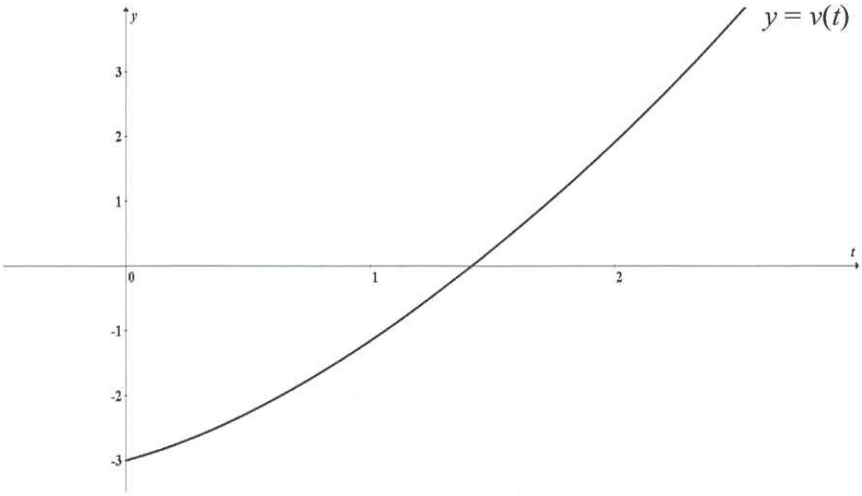
thus $v = t^2 + \sin t - 3$

(1) correct integral

(11) finding c.

[2 marks]

iii. The graph of $y = v(t)$ is shown, in part, below.



Find the distance travelled in the first second of its motion.

$$d = \int_1^0 v(t).dt$$

$$= \int_1^0 t^2 + \sin t - 3 \,.\, dt$$

$$= \left[\frac{t^3}{3} - \cos t - 3t\right]_1^0$$

$$= \cos 1 + \frac{5}{3}$$

(1) integral with correct end points or (-) sign.

(1) ans

[2 marks]

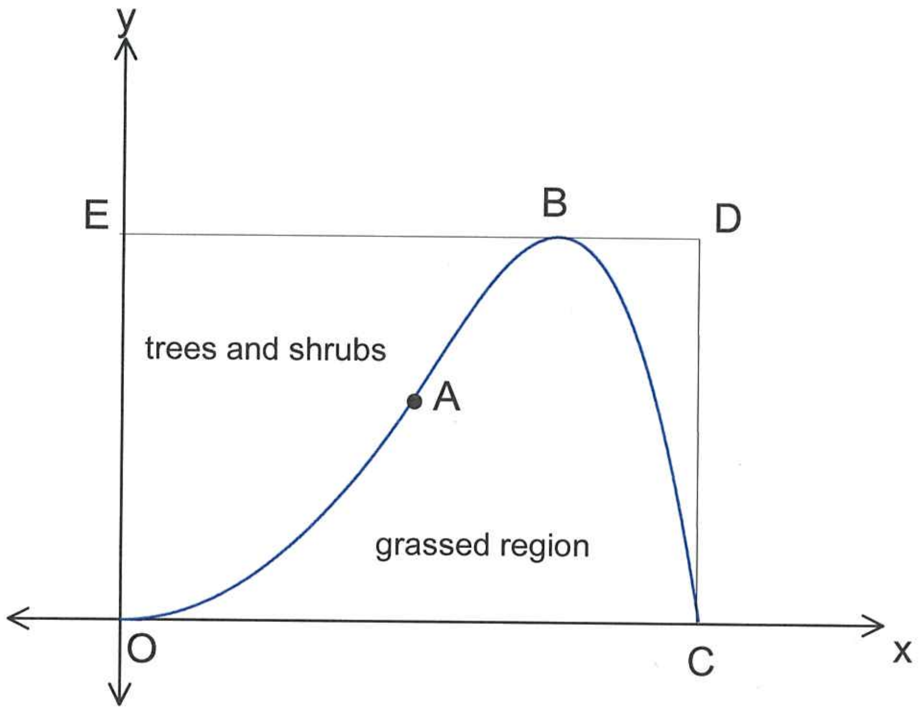
iv. Find $\int_0^3 v\,dt$, and explain what information the answer gives in relation to the train's motion.

$$\int_0^3 v.dt = 1 - \cos(3)$$

- this is train's displacement (position) after 3 seconds

[2 marks]

Question 3.



Euler is a keen gardener and is working on the plan for the landscape of a section of his new back garden. This section of land is a rectangular block OCDE starting from the back line of his house, which lies along the ***x*-axis** and the left side of his fence which is the ***y*-axis**. Distances are measured in metres, OC = 4 metres and the origin O is shown. A path creates a boundary, separating the grassed area (the area between the curve and the ***x*-axis**) from the trees and shrubs, and is made up of two curves, given by the hybrid function

$$f(x) = \begin{cases} x^2 & ,0 \le x \le 2 \\ ax^3 + bx^2 + cx + d & ,2 < x \le 4 \end{cases}$$ where ***a***, ***b***, ***c*** and ***d*** are constants.

Two curves that represent the boundary have the same gradient at the point A, where $x = 2$.

a. **i.** Using the above information **state** the coordinates of points A and C.

A : (2, 4) (1) ans

C : (4, 0) (1) ans

[2 marks]

ii. Hence, explain why it can be stated that:

$8a + 4b + 2c + d = 4$

$64a + 16b + 4c + d = 0$

sub (2,4) into $ax^3 + bx^2 + cx + d = f(x)$.

gives $8a + 4b + 2c + d = 4$ (1) ans

subst. (4,0) into eqn gives

$64a + 16b + 4c + d = 0$ (1) ans

[2 marks]

The x co-ordinate of point B is halfway between the points A and C and is the furthest point reached by boundary and the garden block as measured in the positive y-direction.

iii. **State** the x-coordinate of point B and **find** a third equation in terms of **a**, **b**, and **c** using the above information.

at B, $x = 3$ (1) $x = 3$

this is a max value so $\frac{dy}{dx} = 0$ (1) seeing derivative expression

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

$0 = 27a + 6b + c$ (1) eqn $= 0$

[3 marks]

iv. Use the fact that the two curves have the same gradient at $x = 2$ to write down a fourth equation in terms of **a**, **b**, **c** and **d**.

at $x = 2$ (1) equating derivatives of both eqns

$2x = 3ax^2 + 2bx + c$

and $\therefore 4 = 12a + 4b + c$ (1) correct eqn

[2 marks]

Suppose that $a = -2$, $b = 13$, $c = -24$ and $d = 16$

b. Use definite integrals to write an expression for the total area of the grassed region and hence find the area of the grassed region.

$A = \int_0^2 x^2 . dx + \int_2^4 -2x^3 + 13x^2 - 24x + 16 . dx$

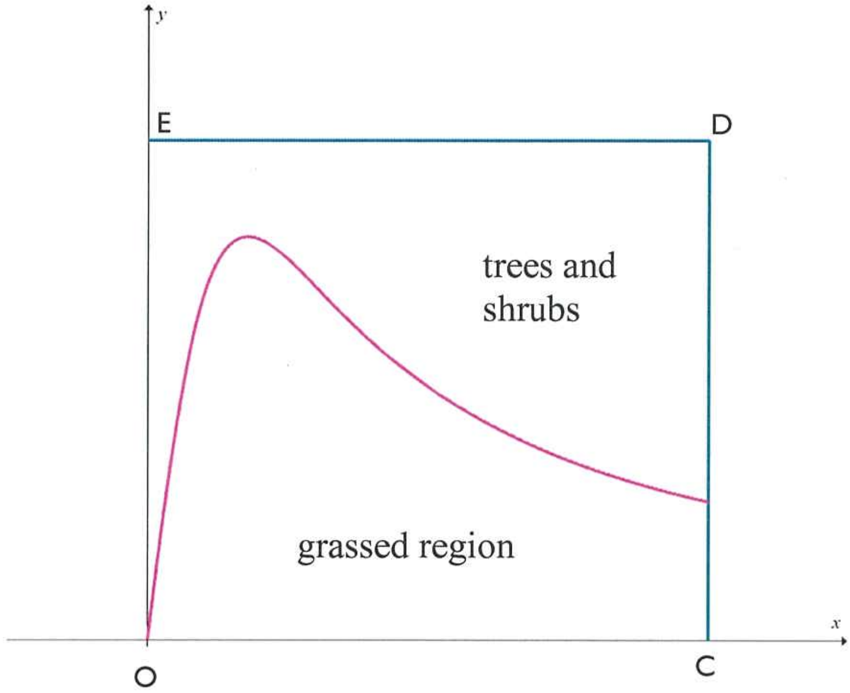
$= \frac{40}{3}$ sq m.

(1) integral

(1) answer

[2 marks]

Euler tries out an alternative plan for the path which gives a different shape for the grassed area.



c. **i.** Differentiate $\log_e(2x^2+1)$ and **hence** evaluate $\int_0^4 \frac{16x}{2x^2+1}dx$

$\frac{d(\ln(2x^2+1))}{dx} = \frac{4x}{2x^2+1}$

(1) correct derivative

(1) correct integral

(1) correct ans

So $4\times\int \frac{4x}{2x^2+1}.dx = 4\times \ln(2x^2+1)$

$\int_0^4 \frac{16x}{2x^2+1}.dx = \left[4\ln(2x^2+1)\right]_0^4$

$= 4\ln 33$

[3 marks]

ii. The equation of the path is given by $g:[0,4]\rightarrow R$, $g(x)=\frac{16x}{2x^2+1}$.

Find the area of the garden occupied by trees and shrubs, correct to 2 decimal places.

at ED, $f(3) = 7$

so area of rectangle $= 7\times 4 = 28$

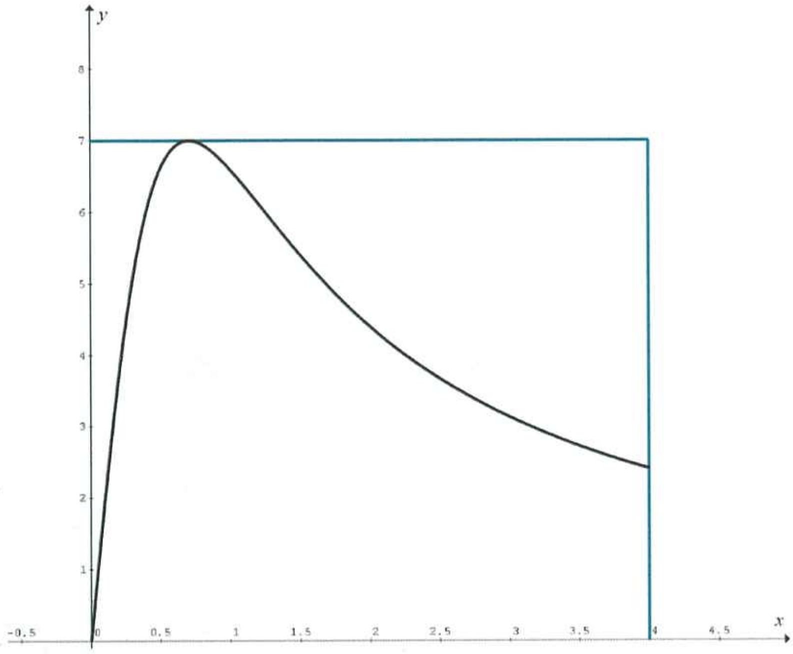
$\therefore$ Area of trees + shrubs $= 28 - 4\ln 33$

$= 14.01$ sq m.

[3 marks]

d. Euler liked the shape of this path but wanted it to go up to the back fence (at $y = 7$) before curving back as shown in the graph below.

If the equation of this path is $h(x) = \dfrac{ax}{2x^2+1}$, $0 \le x \le 4$ and a is a constant, find the value of a.



max value at $y = 7$

$\therefore 7 = \dfrac{ax}{2x^2+1}$ ——— (1)

for max value $h'(x) = 0$

$\therefore 0 = \dfrac{-a(2x^2-1)}{(2x^2+1)^2}$

$x = \pm \dfrac{1}{\sqrt{2}}$ (but $x > 0$)

So $x = \dfrac{1}{\sqrt{2}}$

[4 marks]

sub into (1)

$7 = \dfrac{a \times \frac{1}{\sqrt{2}}}{2\left(\frac{1}{\sqrt{2}}\right)^2 + 1}$

$\therefore a = 14\sqrt{2}$